Improper Integrals

Introduction # Riemann Stieltjes integral $\int_a^b f(x)\,d\alpha$ or Riemann integral $\int_a^b f\,dx$ is defined under the restriction that both f, α are defined and bounded on a finite interval $[a\ b]$. However the symbol $\int_a^b f\,d\alpha$ may sometimes have meaning (ie denote a number), even when f is not bounded or when either a or b or both are infinite. In such cases the symbol

$$\int_a^b f\,d\alpha$$

is called an improper or generalised or infinite integral. Thus the integrals with unbounded integrand or with unbounded interval of integration are improper integrals.

Note # For the sake of distinction an integral which is not improper will be called a a proper integral:-

Improper Integral of First Kind

The integral $\int_a^b f\,d\alpha$ is called improper integral of 1st kind if the integrand remains bounded but integral Integration is unbounded.

Def# Let f, α be defined on $[a, \infty)$. Suppose that $f \in R(\alpha; a, t) = R_\alpha[a, t]$ for every $t \geq a$. keeping a, f, α fixed define a function I on $[a, \infty)$ as

$$I(t) = \int_a^t f(x)\, d\alpha(x) \qquad t \geq a.$$

The function $I(t)$ so defined is called an infinite integral (or an improper integral of 1st kind) and is denoted by $\int_a^\infty f\, d\alpha$.

The integral $\int_a^\infty f\, d\alpha$ is said to converge or said to exist if

$$\lim_{t \to \infty} I(t) = \lim_{t \to \infty} \int_a^t f\, d\alpha \text{ exists (finite)}$$

otherwise $\int_a^\infty f\, d\alpha$ is said to diverge or we say integral does not exist.

If $\lim_{t \to \infty} I(t)$ exists and equals A, then the number A is called value of the integral and we write.

$$\int_a^\infty f(x)\, d\alpha(x) = A$$

Similarly we define the improper integral $\int_{-\infty}^b f\, d\alpha$ as

$$\int_{-\infty}^b f\, d\alpha = \lim_{t \to -\infty} \int_t^b f\, d\alpha$$

Q# Check the Convergence and Divergence of (1) $\int_1^\infty \frac{1}{x}\,dx$ (2) $\int_1^\infty \frac{1}{x^2}\,dx$

Sol (1) $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \left[\ln|x|\right]_1^t$$

$$= \lim_{t\to\infty} \left[\ln t - \ln 1\right]$$

$$= \lim_{t\to\infty} \left[\ln t\right] = \infty$$

$\Rightarrow \int_1^\infty \frac{1}{x}\,dx$ diverges.

(2) $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t x^{-2}\,dx$

$$= -\lim_{t\to\infty} \left[\frac{1}{x}\right]_1^t$$

$$= -\lim_{t\to\infty} \left[\frac{1}{t} - 1\right] = 1$$

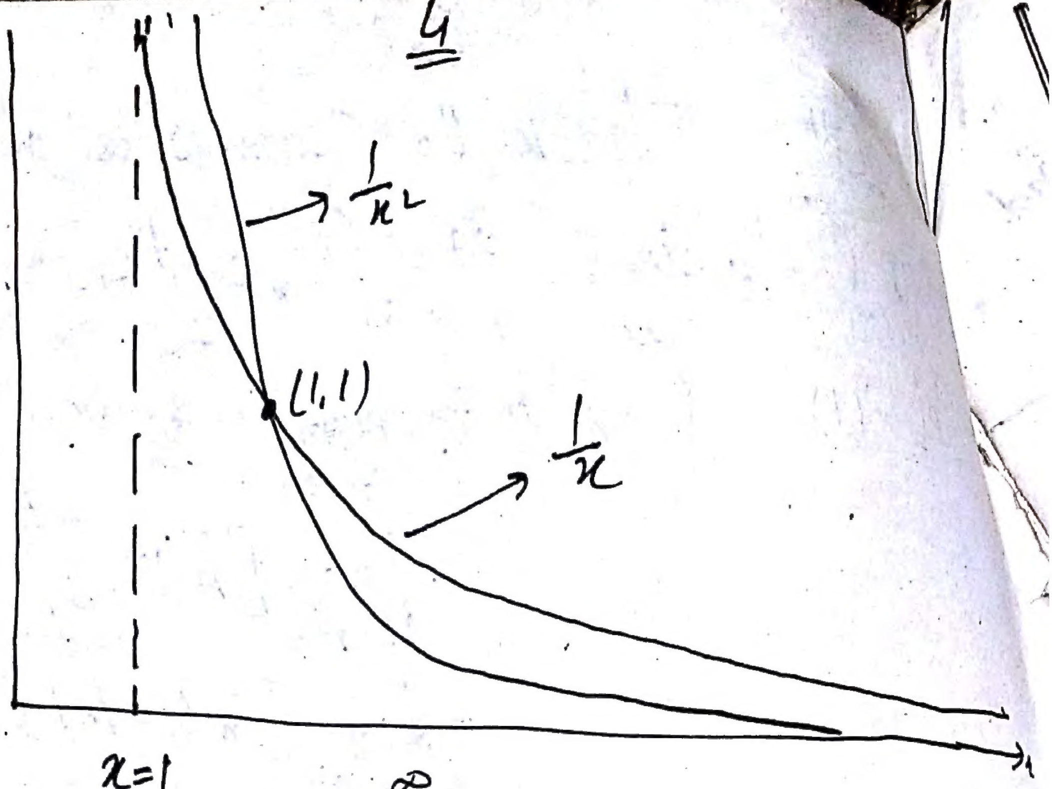
Discussion # we note that $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x^2}$

i.e both functions die as $x \to \infty$ but we also note that

$$\lim_{x\to\infty} \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$\Rightarrow \frac{1}{x^2}$ dies faster than $\frac{1}{x}$ as $x \to \infty$

Therefore $\int_1^\infty \frac{1}{x^2}\,dx$ Converges.



We note that $\int_1^\infty x^2 dx$ diverges because $f(x) = x^2 \longrightarrow \infty$ as $x \to \infty$ ie does not die. Thus when a function does not die its integral does not converge but when a function dies as $x \to \infty$ we may expect its integral of type $\int_a^\infty f d\alpha$ to be convergent and we may equally expect it to be ~~converge~~ divergent as we have seen above in case of integrals $\int_1^\infty \frac{1}{x} dx$ & $\int_1^\infty \frac{1}{x^2} dx$

$$\boxed{\text{Integral } \int_{-\infty}^{+\infty} f(x) d\alpha(x)}$$

If for some $c \in (a\ \infty)$ $\int_{-\infty}^{c} f d\alpha$ and $\int_c^\infty f d\alpha$ are both convergent, then $\int_{-\infty}^{\infty} f d\alpha$ is cgt and its value is defined to be

$$\int_{-\infty}^{\infty} f d\alpha = \int_{-\infty}^{c} f d\alpha + \int_c^\infty f d\alpha$$

Cauchy Principal Value of $\int_{-\infty}^{\infty} f\,d\alpha$

Consider integral $\int_{-\infty}^{1} x\,dx$

The integral $\int_{1}^{\infty} x\,dx$ & $\int_{-\infty}^{1} x\,dx$ both diverge

and hence the integral $\int_{-\infty}^{\infty} x\,dx$ diverges.

But

$\lim_{c \to \infty} \int_{-c}^{c} x\,dx = 0$. It is called cauchy principal value and it may exist even if the integral $\int_{-\infty}^{\infty} f\,d\alpha$ diverges as has been above.

Again $\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \lim_{t \to \infty} \int_{t}^{-1} \frac{1}{x^2} + \int_{-1}^{0} \frac{1}{x^2}\,dx$

$+ \int_{0}^{1} \frac{1}{x^2}\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx$

is divergent but

$$\lim_{c \to \infty} \int_{-c}^{c} \frac{1}{x^2}\,dx = -\lim_{c \to \infty} \left[\frac{1}{x}\right]_{-c}^{c}$$

$$= -\lim_{c \to \infty} \left[\frac{1}{c} + \frac{1}{c}\right] = 0$$

$$\Rightarrow p.v \int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = 0$$

But $\int_{-\infty}^{+\infty} f\,d\alpha$ Converges it Converges to its principal value i.e for a convergent integral value of the integral is same as principal value.

Note # If we know the convergence of integral $\int_{-\infty}^{\infty} f\,d\alpha$ in advance, we may find its value by finding the principal value.

Analogy Between Infinite Integral And Infinite Series

$$\int_a^{\infty} f\,d\alpha \qquad\qquad \sum_{n=1}^{\infty} a_n$$

Here analogy is as

$I(t) = \int_a^t f\,d\alpha$ analogous to $s_n = \sum_{k=1}^{n} a_k$

partial integral　　　　　partial sum.

$\int$ is analogous to $\sum$

$f(x)$ is " " a_n

x corresponds to n

$x \downarrow$ varies continuously on $[a, \infty)$

$n \downarrow$ varies discretely on $\{1, 2, \cdots\}$

Improper Integral of 2nd Kind

If in the definite integral $\int_a^b f\,d\alpha$, interval of integration is finite but f has one or more points of infinite discontinuity i.e f is not bounded on $[a, b]$, then $\int_a^b f\,d\alpha$ is called an improper integral of 2nd kind

e.g $\int_0^1 \frac{dx}{x}$, $\int_1^2 \frac{dx}{2-x}$

Improper Integral of 3rd Kind

If in definite integral $\int_a^b f d\alpha$, the interval is unbounded and f is also un bounded, then it is called improper integral of 3rd kind.

e.g $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}$

Convergence & Divergence of Improper Integral of 2nd kind

(a) Convergence at Left End point

Let f be defined on $(a\ b]$ and integrable (RS or R) on $[t, b]$ $\forall t > a$ or on $[a+\epsilon, b]$, $\epsilon > 0$ or $\forall \epsilon$, $0 < \epsilon < b-a$, then $\int_a^b f d\alpha$ is defined by

$$\int_a^b f d\alpha = \lim_{\epsilon \to 0^+} I(\epsilon) = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f d\alpha.$$

$$= \lim_{t \to a^+} \int_t^b f d\alpha.$$

If this limit exist and is equal to a real number A, then improper integral converges to A otherwise diverges.

<u>Note</u> If $\lim_{x \to a} f(x)$ exists but f is discontinuous at a, then $\int_a^b f d\alpha$ is considered as proper

and proper integral is always Convergent

→ If f is Continuous on $[a\ b]$ except that $f(c^+) \neq f(c^-)$, $a < c < b$ i.e f has a finite jump at c, then $\int_a^b f\,d\alpha$ is considered as proper.

(b) **Convergence at Right End Point #**

Let b be only point of infinite discontinuity and f is defined on (a, b), $f \in R(\alpha)$ on $[a, b)$ or on $[a, t]$ $\forall\, t < b$, $a \le t < b$, then integral $\int_a^b f\,d\alpha$ defined as limit $\int_a^{b-\epsilon} f\,d\alpha$ as

$$\int_a^b f\,d\alpha = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f\,d\alpha \qquad 0 < \epsilon < b-a$$

$$= \lim_{t \to b^-} \int_a^t f\,d\alpha \qquad 0 < t < b-a$$

$$t \in [a, b)$$

If this limit exists, then the integral is cgt, otherwise dgt.

(c) **Convergence at Interior Point**

If an interior point c, $a < c < b$ is the only point of infinite discontinuity (i.e f is unbounded) at c, then

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Integral is cgt if both integrals on R.H.S are Convergent otherwise is dgt.

(d) Convergence At Both End Points

If a & b are both points of infinite discontinuity, then for any c within the interval $[a, b]$

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

The integral exists if both integrals on R.H.S exist otherwise integral doesnot exist.

Example

Discuss the convergence and divergence of f. Integrals

(a) $\int_0^1 \frac{1}{x^p}\,dx$ (b) $\int_1^\infty \frac{1}{x^p}\,dx$ (c) $\int_0^\infty \frac{1}{x^p}$

Sol # (a) Function $f(x) = \frac{1}{x^p}$ is continuous on $(0 \cdot 1]$ irrespective of the value of p but is undefined at $x=0$

Case 1 When $p<0$, f is bounded in $(0\ 1]$, so we can extend the definition to $x=0$ by setting the value of f to be 0 at $x=0$.

If $p=0$ f is identically 1 through out $[0\ 1]$.

Thus for $p \le 0$, f itself has a continuous extension to whole of $[0\ 1]$ and is Riemann integrable there

Case II # If $p>0$, $f(x) = \frac{1}{x^p}$ is unbounded at $x=0$ and integral is improper.

If $0<p<1$, then $1-p>0$ and

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-p}\,dx$$

$$= \lim_{t\to 0^+} \left[\frac{x^{1-p}}{1-p}\right]_t^1$$

$$= \lim_{t\to 0^+} \left(\frac{1}{1-p} - \frac{t^{1-p}}{1-p}\right)$$

$$= \frac{1}{1-p}$$

$\Rightarrow$ Integral Converges to $\frac{1}{1-p}$

<u>Case III</u> # If $p>1$, $1-p<0$ and

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t\to 0^+} \int_t^1 x^{-p}dx$$

$$= \lim_{t\to 0^+} \left[\frac{1}{(1-p)x^{p-1}}\right]_t^1$$

$$= \lim_{t\to 0^+} \left(\frac{1}{1-p} - \frac{1}{(1-p)t^{p-1}}\right)$$

$$= \lim_{t\to 0^+} \left[\frac{1}{1-p} + \frac{1}{(p-1)t^{p-1}}\right]$$

$$= \infty$$

$\Rightarrow$ Improper integral diverges.

<u>Case IV</u> # For $p=1$

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}dx$$

$$= \lim_{t\to 0^+} \left[\ln|x|\right]_t^1$$

$$= \lim_{t\to 0^+} \left[\ln 1 - \ln t\right]$$

$$= +\infty$$

$\Rightarrow$ Integral diverges for $p=1$

Result $\Rightarrow \int_0^1 \frac{1}{x^p}\,dx$ is cgt if $p<1$
dgt if $p \geq 1$

(b) $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx$

$$= \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right)$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ +\infty & \text{if } p<1 \end{cases}$$

For $p=1$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty}\left[\ln|x|\right]_1^t = \lim_{t\to\infty}[\ln t - \ln 1]$$

$$= +\infty$$

Result $\Rightarrow \int_1^\infty \frac{1}{x^p}\,dx$ is cgt if $p>1$
is dgt if $p\leq 1$.

(c) $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx = \int_0^1 \frac{1}{x^p}\,dx + \int_1^\infty \frac{1}{x^p}\,dx$

$I_1 + I_2$

For $p<0$ I_1 is cgt, I_2 is dgt to $+\infty$
For $p=0$ I_1 is cgt, I_2 is dgt to $+\infty$
For $0<p<1$ I_1 is cgt, I_2 is dgt to $+\infty$
For $p=1$ I_1 & I_2 both diverge to $+\infty$

For $p > 1$, I_1 is dgt. to $+\infty$ & I_2 is cgt.

$\Rightarrow$ For any arbitrary p one of the integral diverges to $+\infty$. Hence the integral diverges to $+\infty$ for all p.

Examples

Examine the Convergence and Divergence of

(1) $\int_0^\infty e^{-mx}\,dx \quad (m>0)$ (2) $\int_a^\infty \frac{x}{1+x^2}\,dx$

(iii) $\int_0^\infty \sin x\,dx$ (4) $\int_0^\infty \frac{dx}{(1+x)^3}$

(5) $\int_0^\infty \frac{dx}{x^2+4a^2}$ (6) $\int_3^\infty \frac{dx}{(x-2)^2}$

(7) $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$ (8) $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$

(9) $\int_2^\infty \frac{2x^2}{x^4-1}\,dx$ (10) $\int_1^\infty \frac{x}{(1+2x)^3}\,dx$

Solutions

(1) $$\int_0^\infty e^{-mx}\,dx = \lim_{t\to\infty}\int_0^t e^{-mx}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{e^{-mx}}{-m}\right]_0^t$$

$$= -\frac{1}{m}\lim_{t\to\infty}\left[e^{-mt}-1\right]$$

$$= -\frac{1}{m}[0-1] = \frac{1}{m}, \text{ which is finite.}$$

$\Rightarrow$ Integral converges

Result

$\int_0^\infty e^{-mx}\,dx$ Converges for every $m>0$

(2) $\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\int_a^t \frac{x}{1+x^2}\,dx$

$= \lim_{t\to\infty}\left[\frac{1}{2}\log(1+x^2)\right]_a^t$

$= \lim_{t\to\infty}\frac{1}{2}\left[\log(1+t^2)-\log(1+a^2)\right]$

$= \infty$

$\Rightarrow$ Integral diverges.

(3) $\int_0^\infty \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx$

$= -\lim_{t\to\infty}\left[\cos x\right]_0^t$

$= -\lim_{t\to\infty}\left[\cos t-\cos 0\right]$

$= \lim_{t\to\infty}\left[\cos t-1\right]$, which does not exist because $\cos t$ oscillates between -1 and 1

$\Rightarrow \int_0^\infty \sin x\,dx$ oscillates.

(4) # $\int_0^\infty \frac{dx}{(1+x)^3} = \lim_{t\to\infty}\int_0^t \frac{1}{(1+x)^3}\,dx$

$= \lim_{t\to\infty} -\frac{1}{2}\left[\frac{1}{(1+t)^2}-1\right]$

$= -\frac{1}{2}(0-1) = \frac{1}{2}$

$\Rightarrow$ Integral converges to $\frac{1}{2}$

(5) $\displaystyle\int_0^\infty \frac{dx}{x^2+4a^2} = \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+(2a)^2}$

$\displaystyle = \lim_{t\to\infty}\left[\frac{1}{2a}\tan^{-1}\frac{x}{2a}\right]_0^t$

$\displaystyle = \lim_{t\to\infty}\frac{1}{2a}\left[\tan^{-1}\frac{t}{2a} - \tan^{-1}0\right]$

$\displaystyle = \frac{1}{2a}\left[\frac{\pi}{2}\right] = \frac{\pi}{4a}$, which is finite

$\Rightarrow$ Integral converges to $\frac{\pi}{4a}$

(6) $\displaystyle\int_0^\infty e^{2x}\,dx = \lim_{t\to\infty}\int_0^t e^{2x}\,dx$

$\displaystyle = \lim_{t\to\infty}\frac{1}{2}\left[e^{2x}\right]_0^t$

$\displaystyle = \lim_{t\to\infty}\frac{1}{2}\left[e^{2t} - e^{0}\right]$

$= \infty$

$\Rightarrow$ Integral diverges to $+\infty$

Resut

Note that $\displaystyle\lim_{x\to\infty} e^{2x} = \infty$ i.e. the integrand does not die as $x \to \infty$ so integral diverges.

On the other hand $\displaystyle\int_0^\infty e^{-mx}$ converges for all $m>0$. Here $\displaystyle\lim_{x\to\infty} e^{-mx} = 0$ $\forall\, m>0$. So we may expect convergence which comes out.

* Knowledge non-negative and bounded is a great blessing of God.

(7) # $$\int_3^\infty \frac{dx}{(x-2)^2} = \lim_{t\to\infty} \int_0^t (x-2)^{-2}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{(x-2)^{-1}}{-1}\right]_0^t$$

$$= -\lim_{t\to\infty}\left[\frac{1}{t-2} - 1\right] = -(0-1) = 1$$

$\Rightarrow$ Integral Converges.

(8) # $$\int_0^\infty \frac{dx}{(1+x)^{2/3}} = \lim_{t\to\infty} \int_0^t (1+x)^{-2/3}\,dx$$

$$= \lim_{t\to\infty} 3\left[(1+x)^{1/3}\right]_0^t$$

$$= 3\lim_{t\to\infty}\left[(1+t)^{1/3} - 1\right]$$

$$= +\infty$$, Divergent

(9) # $$\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty}\left[\sec^{-1}x\right]_{\sqrt{2}}^t$$

$$= \lim_{t\to\infty}\left[\sec^{-1}t - \sec^{-1}\sqrt{2}\right]$$

$$= \frac{\pi}{2} - \pi/4 = \frac{\pi}{4}$$ which is finite.

$\Rightarrow$ Integral is cgt.

(10) # $$\int_2^\infty \frac{2x^2}{x^4-1}\,dx = \lim_{t\to\infty}\int_2^t \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)}\,dx$$

$$= \lim_{t\to\infty}\int_2^t\left[\frac{1}{x^2-1} + \frac{1}{x^2+1}\right]dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + \tan^{-1}x\right]_2^t$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\log\frac{t-1}{t+1} + \tan^{-1}t - \frac{1}{2}\log\frac{1}{3} - \tan^{-1}2\right]$$

$$= \frac{1}{2}\lim_{t\to\infty}\log\left(\frac{1-\frac{1}{t}}{1+1/t}\right) + \pi/2 + \frac{1}{2}\log 3 - \tan^{-1}2$$

$$= \frac{1}{2}\log 1 + \pi/2 + \frac{1}{2}\log 3 - \tan^{-1}2$$

$$= \frac{\pi}{2} + \frac{1}{2}\log 3 - \tan^{-1}2 \quad \text{which is finite.}$$

$\Rightarrow$ Integral Converges.

(11) # $\displaystyle\int_1^\infty \frac{x}{(1+2x)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{(1+2x)^3}\,dx$

$$= \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(1+2x)-\frac{1}{2}}{(1+2x)^3}\,dx$$

$$= \lim_{t\to\infty}\int_1^t\left[\frac{1}{2}(1+2x)^{-2} - \frac{1}{2}(1+2x)^{-3}\right]dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\frac{(1+2x)^{-1}}{-1\times 2} - \frac{1}{2}\frac{(1+2x)^{-2}}{-2\times 2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2x)} + \frac{1}{8(1+2x)^2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2t)} + \frac{1}{8(1+2t)^2} + \frac{1}{12} - \frac{1}{72}\right]$$

$$= 0 + 0 + \frac{1}{12} - \frac{1}{72} = \frac{5}{72} \quad \text{which is finite.}$$

$\Rightarrow$ Integral Converges.

Next Do yourself.

Examples#

Examine the Convergence or Divergence of following integrals.

(1) $\int_1^\infty x\,e^{-x}\,dx$ (2) $\int_0^\infty x^2 e^{-x}\,dx$

(3) $\int_0^\infty x\,e^{-x^2}\,dx$ (4) $\int_0^\infty x^3 e^{-x^2}\,dx$

(5) $\int_0^\infty x\sin x\,dx$

Solution# (1) $\int_1^\infty x\,e^{-x}\,dx = \lim_{t\to\infty}\int_1^t x\,e^{-x}\,dx$

$= \lim_{t\to\infty}\left[-xe^{-x}-e^{-x}\right]_1^t = \lim_{t\to\infty}\left(-te^{-t}-e^{-t}+e^{-1}+e^{-1}\right)$

$= \lim_{t\to\infty}\left(-e^{-t}t\right) - \lim_{t\to\infty} e^{-t} + \frac{2}{e}$

$= 0 + 2/e = 2/e$ which is finite.

$\Rightarrow$ Integral Converges

(2) $\int_0^\infty x^2 e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2 e^{-x}\,dx$

$= \lim_{t\to\infty}\left[-x^2e^{-x}-2xe^{-x}-2e^{-x}\right]_0^t$

$= 2$, which is finite

(3) $\int_0^\infty x\,e^{-x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}\int_0^t -2x\,e^{-x^2}\,dx$

$= -\frac{1}{2}\lim_{t\to\infty}\left[e^{-x^2}\right]_0^t$

$= -\frac{1}{2}\lim_{t\to\infty}\left(e^{-t^2}-e^0\right) = -\frac{1}{2}(0-1) = \frac{1}{2}$

(4)# $\int_0^\infty x^3 e^{-x^2} dx = \lim_{t\to\infty} \int_0^t x \cdot x^2 e^{-x^2} dx$

$= \lim_{t\to\infty} \left[-\frac{1}{2} x e^{-x^2} \Big|_0^t + \int_0^t \frac{1}{2} x e^{-x^2} dx \right]$

Let $x^2 = z$ $\quad 2x\,dx = dz$

$x\,dx = \frac{1}{2} dz$

Limits When $x = 0 \quad z = 0$

$x = \infty \quad z = \infty$

$\int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2} \int_0^\infty z e^{-z} dz$

$= \frac{1}{2} \lim_{t\to\infty} \int_0^t z e^{-z} dz$

$= \frac{1}{2} \lim_{t\to\infty} \left[-z e^{-z} - e^{-z} \right]_0^t$

$= \frac{1}{2} \lim_{t\to\infty} \left[-t e^{-t} - e^{-t} - 0 + e^0 \right]$

$= \frac{1}{2} [0 + 0 + 1] = \frac{1}{2}$, which is finite.

$\Rightarrow$ Integral Converges

(5) $\int_0^\infty x \sin x\, dx = \lim_{t\to\infty} \int_0^t x \sin x\, dx$

$= \lim_{t\to\infty} \left[-x\cos x + \sin x \right]_0^t$

$= \lim_{t\to\infty} \left(-t\cos t + \sin t \right)$

which oscillates b/w $+\infty, -\infty$ because $\cos t$ oscillates b/w -1 and $+1$ as $t\to\infty$

$\Rightarrow$ Integral oscillates infinitely.